



MathsWithDan – Differentiation – The Chain Rule

Starter

Differentiate:

1. $y = 4x^3 + x^2 + 5x + 1$ [2 Marks]

$$\frac{dy}{dx} = 12x^2 + 2x + 5$$

2. $y = (2x + 1)^3$ [3 Marks]

$$= (2x + 1)(2x + 1)(2x + 1)$$

$$= (4x^2 + 4x + 1)(2x + 1)$$

$$= 8x^3 + 12x^2 + 6x + 1$$

$$\frac{dy}{dx} = 24x^2 + 24x + 6$$



Composite Functions (2 Marks)

It is given that

$$f(x) = x^3 \quad \text{and} \quad g(x) = x + 1$$

Find

$$f(g(x)).$$

$$f(g(x)) = (x + 1)^3$$

$$\boxed{f(g(x)) = (x + 1)^3}$$

Challenge

How would you differentiate

$$y = (2x + 1)^8 ?$$

Have a think about it before we begin the lesson.

The Chain Rule

Key Idea

The Chain Rule is used to differentiate composite functions (also known as functions of another function).

A composite function is created when one function is inside another.

For example:

$$(2x + 1)^5, \sin(3x), (x^2 + 4)^7$$

In each case, there is an outer function and an inner function.

The Chain Rule

If

$$y = f(u)$$

and

$$u = g(x),$$

then

$$\boxed{\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx}}$$

where:

- $\frac{dy}{du}$ is the derivative of the outer function.
 - $\frac{du}{dx}$ is the derivative of the inner function.
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Key Steps

When differentiating using the Chain Rule:

1. Identify the inner function.
2. Differentiate the outer function, keeping the inside unchanged.
3. Multiply by the derivative of the inner function.
4. Simplify your final answer where appropriate.

Worked Example 1

Given that $y = (2x^3 - 5x)^6$, find $\frac{dy}{dx}$ using the Chain Rule.

Working

Let $u = 2x^3 - 5x$

$$\frac{du}{dx} = 6x^2 - 5$$

$$y = u^6$$

$$\frac{dy}{du} = 6u^5$$

Using the chain rule,

$$\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx}$$

$$= 6u^5 (6x^2 - 5)$$

$$\frac{dy}{dx} = 6(2x^3 - 5x)^5 (6x^2 - 5)$$

Differentiate u with respect to x to get $\frac{du}{dx}$.

Substitute u into the equation for y and differentiate with respect to u to get $\frac{dy}{du}$.

Use the chain rule to find $\frac{dy}{dx}$ in terms of u .

Use $u = 2x^3 - 5x$ to write your final answer in terms of x only.

Answer: $6(2x^3 - 5x)^5 (6x^2 - 5)$

Worked Example 2

Given that $y = \cos^4(x)$, find $\frac{dy}{dx}$ using the Chain Rule.

Working

$$y = \cos^4 x = (\cos x)^4$$

Let $u = \cos x$:

$$\frac{du}{dx} = -\sin x$$

$$y = u^4$$

$$\frac{dy}{du} = 4u^3$$

Using the chain rule,

$$\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx}$$

$$= 4u^3(-\sin x)$$

$$\frac{dy}{dx} = 4(\cos x)^3(-\sin x)$$

Differentiate u with respect to x to get $\frac{du}{dx}$.

Substitute u into the equation for y and differentiate with respect to u to get $\frac{dy}{du}$.

Use the chain rule to find $\frac{dy}{dx}$ in terms of u .

Substitute $u = \cos x$ back into $\frac{dy}{dx}$ to get an answer in terms of x only.

Answer: $4 \cos^3 x (-\sin x)$

You can write the Chain Rule using function notation:

- The Chain Rule enables you to differentiate a function of a function. In general,
 - if $y = (f(x))^n$ then $\frac{dy}{dx} = n(f(x))^{n-1} f'(x)$
 - if $y = f(g(x))$ then $\frac{dy}{dx} = f'(g(x)) g'(x)$

Worked Example 3

Given that $y = \sqrt{3x^2 + 7}$, find $\frac{dy}{dx}$ at $(2, 19)$.

Working

Let $u = 3x^2 + 7$

$$\frac{du}{dx} = 6x$$

$$y = u^{\frac{1}{2}}$$

$$\frac{dy}{du} = \frac{1}{2} u^{-\frac{1}{2}}$$

Using the chain rule,

$$\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx}$$

$$= \frac{1}{2} u^{-\frac{1}{2}} (6x)$$

Substitute $u = 3x^2 + 7$ to get

$$\frac{dy}{dx} = \frac{3x}{\sqrt{3x^2 + 7}}$$

At $(2, \sqrt{19})$, $x = 2$

$$\frac{dy}{dx} = \frac{3(2)}{\sqrt{3(2)^2 + 7}} = \frac{6}{\sqrt{19}}$$

Let u be the inside of the square root. Differentiate u with respect to x to get $\frac{du}{dx}$.

Write y in terms of u and differentiate with respect to u to get $\frac{dy}{du}$.

Use the chain rule to find $\frac{dy}{dx}$ in terms of u .

Substitute $u = 3x^2 + 7$ back into $\frac{dy}{dx}$ to get an answer in terms of x only.

Substitute $x = 2$ to find $\frac{dy}{dx}$ at $(2, \sqrt{19})$.

Answer: $\frac{6}{\sqrt{19}}$



Moderate Questions

a) Differentiate

Let

$$u = 3x - 2$$

Then

$$y = u^5$$

Differentiate y with respect to u :

$$\frac{dy}{du} = 5u^4$$

Differentiate u with respect to x :

$$\frac{du}{dx} = 3$$

Using the Chain Rule,

$$\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx}$$

$$\frac{dy}{dx} = 5u^4 \times 3$$

$$\frac{dy}{dx} = 15u^4$$

Substituting $u = 3x - 2$,

$$\boxed{\frac{dy}{dx} = 15(3x - 2)^4}$$



b) Differentiate

$$y = \frac{1}{5 - 4x}$$

Rewrite as

$$y = (5 - 4x)^{-1}$$

$$\text{Let } u = 5 - 4x$$

$$\text{Then } y = u^{-1}$$

Differentiate y with respect to u :

$$\frac{dy}{du} = -u^{-2}$$

Differentiate u with respect to x :

$$\frac{du}{dx} = -4$$

Using the Chain Rule,

$$\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx}$$

$$\frac{dy}{dx} = (-u^{-2})(-4)$$

$$\frac{dy}{dx} = 4u^{-2}$$

Substituting $u = 5 - 4x$,

$$\frac{dy}{dx} = 4(5 - 4x)^{-2}$$

$$\boxed{\frac{dy}{dx} = \frac{4}{(5 - 4x)^2}}$$



c) Differentiate

$$y = \sqrt{2x + 9}$$

Rewrite as

$$y = (2x + 9)^{\frac{1}{2}}$$

Let

$$u = 2x + 9$$

Then

$$y = u^{\frac{1}{2}}$$

Differentiate y with respect to u :

$$\frac{dy}{du} = \frac{1}{2} u^{-\frac{1}{2}}$$

Differentiate u with respect to x :

$$\frac{du}{dx} = 2$$

Using the Chain Rule,

$$\begin{aligned} \frac{dy}{dx} &= \frac{dy}{du} \times \frac{du}{dx} \\ \frac{dy}{dx} &= \frac{1}{2} u^{-\frac{1}{2}} \times 2 \\ \frac{dy}{dx} &= u^{-\frac{1}{2}} \end{aligned}$$

Substituting $u = 2x + 9$,

$$\frac{dy}{dx} = (2x + 9)^{-\frac{1}{2}}$$

$$\boxed{\frac{dy}{dx} = \frac{1}{\sqrt{2x + 9}}}$$

The following particular case of the Chain Rule is useful for differentiating functions that are not in the form $y = f(x)$.

- $\frac{dy}{dx} = \frac{1}{\frac{dx}{dy}}$

Hint: This is because

$$\frac{dy}{dx} \times \frac{dx}{dy} = \frac{dy}{dy} = 1.$$

Worked Example 4

Find the value of $\frac{dy}{dx}$ at the point $(3,1)$ on the curve with equation $y^3 + 2y = x$.

Working

Start with $x = y^3 + y$.

Differentiate both sides with respect to y .

$$\frac{dx}{dy} = \frac{d}{dy}(y^3) + \frac{d}{dy}(y)$$

$$\frac{dx}{dy} = 3y^2 + 1$$

Substitute $y = 1$.

$$\frac{dx}{dy} = 3(1)^2 + 1 = 4$$

Use $\frac{dy}{dx} = \frac{1}{\frac{dx}{dy}}$.

$$\frac{dy}{dx} = \frac{1}{4}$$

Start with $x = y^3 + y$.

Differentiate with respect to y .

Differentiate y^3 to get $3y^2$ and differentiate y to get 1.

Substitute $y = 1$ (from the point $(2, 1)$) to find $\frac{dx}{dy}$.

Use $\frac{dy}{dx} = \frac{1}{\frac{dx}{dy}}$ to find $\frac{dy}{dx}$.

Answer: $\frac{1}{4}$



★ Exam Challenge

The curve C has equation

$$y = (x^2 - 4)^3.$$

Use calculus to determine the set of values of x for which the curve is increasing.

Give your answer in set notation.

(7 marks)

Mark Scheme

Differentiate using the Chain Rule.

Let

$$u = x^2 - 4.$$

Then

$$y = u^3.$$

Differentiate y with respect to u :

$$\frac{dy}{du} = 3u^2.$$

Differentiate u with respect to x :

$$\frac{du}{dx} = 2x.$$

Using the Chain Rule,

$$\begin{aligned} \frac{dy}{dx} &= \frac{dy}{du} \times \frac{du}{dx} \\ \frac{dy}{dx} &= 3u^2(2x) \\ \frac{dy}{dx} &= 6xu^2. \end{aligned}$$

Substituting $u = x^2 - 4$,



$$\frac{dy}{dx} = 6x(x^2 - 4)^2.$$

M1 A1

To determine where the curve is increasing, we need to find where

$$\frac{dy}{dx} > 0.$$

Notice that

$$(x^2 - 4)^2 \geq 0$$

for every value of x . This is because any number squared is always positive or zero. Therefore, the squared term can never make the derivative negative.

This means that the sign of

$$\frac{dy}{dx}$$

depends only on the sign of the $6x$ term.

- If $x < 0$, then $6x < 0$, so $\frac{dy}{dx} < 0$.
- If $x > 0$, then $6x > 0$, so $\frac{dy}{dx} > 0$.

B1

However, we must also consider when the gradient is equal to zero. Since we are solving the inequality

$$\frac{dy}{dx} > 0,$$

any values of x for which

$$\frac{dy}{dx} = 0$$



must be excluded.

The derivative is zero when

$$6x(x^2 - 4)^2 = 0.$$

Hence,

$$x = 0, x = -2, x = 2.$$

The values $x = -2$ and $x = 0$ are already excluded since they do not satisfy the condition $x > 0$. However, $x = 2$ is positive, so it must also be excluded because the gradient is zero there, not positive.

Therefore,

$$\{x \in \mathbb{R}: x > 0, x \neq 2\}$$

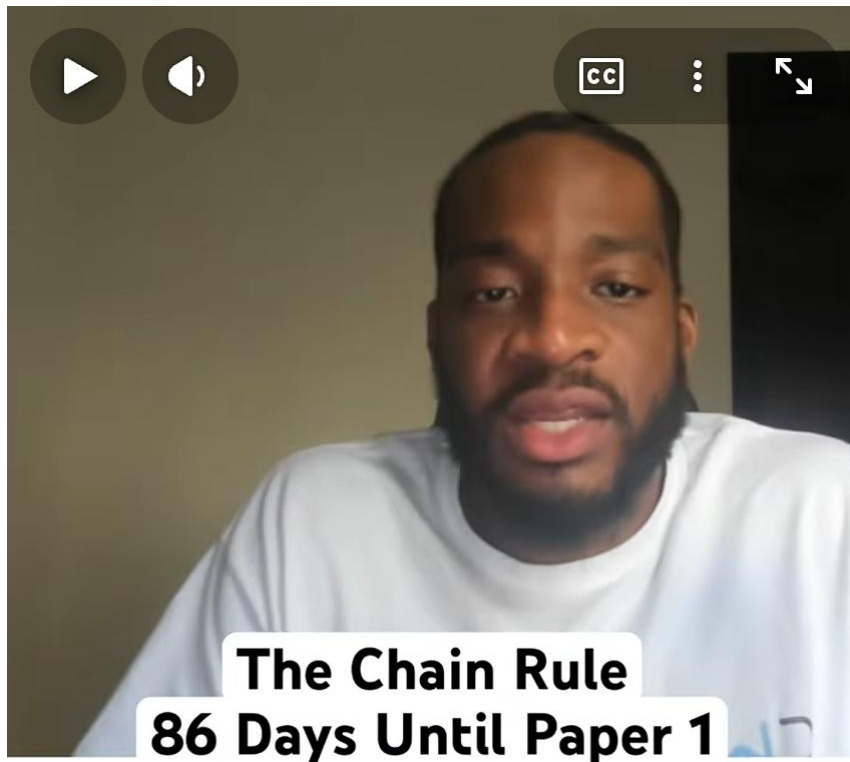
or equivalently,

$$\{x \in \mathbb{R}: 0 < x < 2 \text{ or } x > 2\}.$$

M1 A1 B1 A1

[7 Marks]

 Watch Me Solve It



Use the chain rule to find $\frac{dy}{dx}$ when $y = (x^3 + 5x)^7$

[2 marks]



 Try it first on the next page and then Scan the QR code to watch the full solution.



Use the chain rule to find $\frac{dy}{dx}$ when $y = (x^3 + 5x)^7$. [2 marks]

WOW, DONE ALREADY?! THAT WAS QUICK! SEE YOU SOON!



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